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A "K-Attainable" Inequality Related to the Convergence of Positive Linear Operators*

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The convergence of a sequence of positive linear operators to the identity operator was studied deeply by Korovkin [5]. His main theorem was put in an inequality form by Shisha and Mond [9], measuring the degree of this convergence. Later other authors gave similar, more general/flexible, quantitative results, e.g., Mond [7].

By Riesz representation theorem, the above convergence is closely related to the weak convergence of a sequence of finite measures, to the unit (Dirac) measure at a fixed point.

Introducing a new variation of the K-functional and using the terminology of moments (see [3, 4]), for convenience we consider only probability measures, and we establish a sharp inequality which estimates the degree of the pointwise convergence of a sequence of positive linear operators to the identity operator, all acting on $C([a, b])$, where $[a, b] \subset \mathbb{R}$.

DEFINITION 1. Let $f \in C([a, b])$. Let $n \in \mathbb{N}$ and $x_0 \in [a, b]$ be fixed. We define the *reduced K-functional* as

$$K_n^{(x_0)}(f, t) = \inf(\|f - g\| + t\,\|g^{(n)}\|).$$

Here, $t \geqslant 0$ while g ranges through the functions $g \in C^n([a, b])$ satisfying $g^{(k)}(x_0) = 0$; $k = 1, \ldots, n-1$ where $\|\cdot\|$ denotes the sup-norm.

It is similar to a functional $K_n(f, t)$ employed by Peetre [8], where one does not impose the conditions $g^{(k)}(x_0) = 0$.

The quantity $K_n^{(x_0)}$ measures how well f can be approximated by a smooth function g. It is the right sort of measure for the approximation of small functions f possessing large derivatives in sup-norm.

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Remark 2. The reduced K-functional has the following properties. Namely, it is:

(i) Subadditive in terms of f.

(ii) Continuous, nonnegative, monotonely increasing, and concave in t (and hence subadditive).

(iii) $K_n^{(x_0)}(f, 0) = 0$.

That is, it has the basic properties of the usual Peetre functional $K_n(f, t)$. When $n = 1$, then $K_1^{(x_0)} = K_1$. Obviously

$$K_n(f, t) \leqslant K_n^{(x_0)}(f, t)$$

with equality when f is a constant function.

Now we give the main result of this note:

THEOREM 3. *Let μ be a probability measure on $[a, b]$ with prescribed absolute moment*

$$d_1(x_0) = \int |y - x_0| \, \mu(dy), \tag{3.1}$$

where x_0 is a fixed point of $[a, b]$. Consider $f \in C([a, b])$.

Then

$$\left| \int f \, d\mu - f(x_0) \right| \leqslant 2K_n^{(x_0)} \left(f; \frac{d_1(x_0)(c(x_0))^{n-1}}{2n!} \right), \tag{3.2}$$

where $c(x_0) = \max(x_0 - a, b - x_0)$, $n \in \mathbb{N}$.

The above inequality is attained (i.e., it is sharp) by the function $f(x) = (x - x_0)^n$ and the probability measure μ_0 carried by $\{x_0, a\}$ if $x_0 - a \geqslant b - x_0$, as well as by $\{x_0, b\}$ if $x_0 - a \leqslant b - x_0$: in each case having mass $[1 - (d_1(x_0)/c(x_0))]$ and $d_1(x_0)/c(x_0)$, respectively.

Proof. For $f \in C([a, b])$ and $g \in C^n([a, b])$ we have $f(y) - f(x_0) = (f(y) - g(y)) + (g(y) - g(x_0)) + (g(x_0) - f(x_0))$. Integrating relative to μ obtain

$$\begin{aligned} \left| \int f \, d\mu - f(x_0) \right| &\leqslant \left| \int (f(y) - g(y)) \, \mu(dy) \right| + |g(x_0) - f(x_0)| \\ &\quad + \left| \int (g(y) - g(x_0)) \, \mu(dy) \right| \\ &\leqslant 2 \, \|f - g\| + \int |g(y) - g(x_0)| \, \mu(dy), \end{aligned}$$

where $\| \cdot \|$ denotes the supremum norm.

Now assume that $g^{(k)}(x_0)=0$, $k=1,\ldots,n-1$. By Taylor's theorem we have $\gamma\in(x_0, y)$:

$$|g(y)-g(x_0)|=\left|\frac{g^{(n)}(\gamma)}{n!}(y-x_0)^n\right|\leqslant\frac{\|g^{(n)}\|}{n!}|y-x_0|^n.$$

Thus

$$\int|g(y)-g(x_0)|\,\mu(dy)\leqslant\frac{\|g^{(n)}\|}{n!}\int|y-x_0|^n\,\mu(dy).$$

Letting $d_n(x_0)=(\int|y-x_0|^n\,\mu(dy))^{1/n}$ we then have

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant 2\,\|f-g\|+\frac{\|g^{(n)}\|}{n!}d_n^n(x_0).$$

Now by applying Definition 1 we conclude:

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant 2K_n^{(x_0)}\left(f;\frac{d_n^n(x_0)}{2n!}\right).$$

Since clearly $d_n^n(x_0)\leqslant d_1(x_0)(c(x_0))^{n-1}$ we have proved (3.2).

Furthermore, consider $f(x)=(x-x_0)^n$, which satisfies $f^{(i)}(x_0)=0$; $i=1,\ldots,n-1$. Then, integrating relative to the probability measure μ_0 described in the theorem, one easily verifies the last assertion. Note that (taking $g=f$ in the definition of $K_n^{(x_0)}$) $K_n^{(x_0)}(f,t)\leqslant t\,\|f^{(n)}\|=tn!$. ∎

Corollary 4. *We get the attainable inequality*

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant 2K_1\left(f;\frac{d_1(x_0)}{2}\right)\tag{4.1}$$

(*where K_1 is the usual K-functional*).

Using the terminology of positive linear operators one obtains

Corollary 5 (Pointwise Approximation). *Let $f\in C([a,b])$ and let L be a positive linear operator acting on $C([a,b])$ satisfying*

$$L(1,x)=1\qquad\text{all}\quad x\in[a,b].$$

Then, we have the attainable (i.e., sharp) inequality

$$|L(f,x)-f(x)|\leqslant 2K_1(f;\tfrac{1}{2}L(|y-x|,x)),\tag{5.1}$$

all $x\in[a,b]$.

Proof. By Riesz representation theorem there exists a probability measure μ_x such that $L(f, x) = \int f(t)\, \mu_x(dt)$. And then (5.1) is just another of writing (3.2) with $n = 1$. ∎

As an application of the last we give:

EXAMPLE 6. Let $f \in C([0, 1])$ and consider the Bernstein polynomials (see [6]):

$$(B_N f)(x) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x^k (1-x)^{N-k}, \qquad x \in [0, 1],\ N \in \mathbb{N}.$$

Then by applying (5.1) and using Schwarz's inequality to estimate the argument in the K-functional, we obtain

$$|(B_N f)(x) - f(x)| \leqslant 2K_1\left(f; \frac{1}{2}\sqrt{\frac{x(1-x)}{N}}\right) \leqslant 2K_1\left(f; \frac{1}{4\sqrt{N}}\right).$$

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